

Population-only decay map for n -qubit n -partite inseparability detection

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We introduce a new positive linear map for a single qubit. This map is a decay only in populations of a single-qubit density operator. It is shown that an n -fold product of this map may be used for a detection of n -partite inseparability of an n -qubit density operator (*i.e.*, detection of impossibility of representing a density operator in the form of a convex combination of products of density operators of individual qubits). This product map is also investigated in relation to a variant of the entanglement detection method mentioned by Laskowski and Żukowski.

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Detection of entanglement in a mixed state is of public interest. There have been many methods proposed, such as the partial transpose criterion [1], the range criterion [2], the reduction criterion [3, 4], the majorization criterion [5], the cross norm criterion [6], the matrix realignment criterion [7], the entanglement witness criterion [8, 9], etc. (There are many review articles on these subjects, *e.g.*, Refs. [10, 11, 12].)

Entanglement of a density operator ρ is defined as inseparability of ρ . For a bipartite system consisting of subsystems A and B , ρ is called separable iff there is a decomposition [13]

$$\rho = \sum_i p_i \rho_i^{[A]} \otimes \rho_i^{[B]} \quad (1)$$

using non-negative probabilities p_i ($\sum_i p_i = 1$). Here, variable i is a label. When this decomposition is impossible, ρ is called inseparable.

A *multiseparable* [14] state (or *fully-separable* [15] state) of n qubits is represented by a density operator that can be decomposed in the form of

$$\rho = \sum_i p_i \rho_i^{[1]} \otimes \cdots \otimes \rho_i^{[n]} \quad (2)$$

using non-negative probabilities p_i ($\sum_i p_i = 1$). Any density operator that is not multiseparable is called an *n -partite inseparable* density operator of n qubits or an *n -inseparable* density operator of n qubits. We use the term *n -qubit n -partite inseparability* in this meaning.

To investigate the inseparability, positive but not completely positive (PnCP) linear maps are useful [3, 8, 16]. A positive linear map Λ is a map such that $\sigma \geq 0 \implies \Lambda(\sigma) \geq 0$ for a density operator σ ($\sigma \geq 0$ means $\langle v|\sigma|v \rangle \geq 0$ for $\forall |v\rangle$ in the Hilbert space \mathcal{H} .) For a bipartite system whose Hilbert space is $\mathcal{H}_A \otimes \mathcal{H}_B$, if we have a density operator ρ in the form of Eq. (1),

$(I^{[A]} \otimes \Lambda^{[B]})(\rho) = \sum_i p_i \rho_i^{[A]} \otimes \Lambda(\rho_i^{[B]})$ is kept positive. By the contraposition of this relation, if $(I^{[A]} \otimes \Lambda^{[B]})(\rho)$ is non-positive, then ρ is inseparable [8]. Of course, this is true also when the map $(\Lambda^{[A]} \otimes I^{[B]})$ is used instead of $(I^{[A]} \otimes \Lambda^{[B]})$.

A similar argument is possible for an n -qubit n -partite density operator. When a density operator ρ is in the form of Eq. (2), any positive linear map Λ acting on a single qubit preserves the positivity of $(I \otimes \Lambda)(\rho)$. Here, I is the identity map acting on the rest of the qubits. When $(I \otimes \Lambda)(\rho)$ is a non-positive operator, then ρ is not multiseparable.

Among such PnCP maps, there are two well-known maps: one is a transpose Λ_T of a matrix, $[\Lambda_T(\sigma)]_{ij} = \sigma_{ji}$ [1]; the other one is the map [3] $\Lambda_H(\sigma) = \mathbb{1} \text{Tr} \sigma - \sigma$. These maps have been used for the study of entanglement in bipartite systems intensively. PnCP maps have been investigated also for multipartite systems, *e.g.*, the depolarizing map [17].

Related to the general multiseparability criterion, recently, one important theorem was mentioned by Laskowski and Żukowski [18]:

The Laskowski-Żukowski theorem. *Given a k -partite k -separable density operator, the absolute value of an antidiagonal element is less than or equal to $(1/2)^k$.*

As they claimed that this is obvious, the proof is very simple:

Proof. Let us write the Hilbert space of a k -partite system as $\mathcal{H}^{[P_1]} \otimes \cdots \otimes \mathcal{H}^{[P_k]}$ by using those for subsystems, $\mathcal{H}^{[P_j]}$ ($j = 1, \dots, k$). Consider a k -separable density operator $\rho_{\text{sep}}^{[P_1, \dots, P_k]} = \sum_i p_i \rho_i^{[P_1]} \otimes \cdots \otimes \rho_i^{[P_k]}$ with $p_i \geq 0$ such that $\sum_i p_i = 1$. Any density operator has antidiagonal elements whose absolute values are less than or equal to $1/2$. Hence, for $\forall i$, $\rho_i^{[P_1]} \otimes \cdots \otimes \rho_i^{[P_k]}$ has antidiagonal elements less than or equal to $(1/2)^k$. Thus $\rho_{\text{sep}}^{[P_1, \dots, P_k]}$ has antidiagonal elements less than or equal to $(1/2)^k$. \square

By the contraposition of the relation in this theorem, we have the corollary:

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The Laskowski-Żukowski corollary. *Given a k -partite density operator, if it has an antidiagonal element whose absolute value is larger than $(1/2)^k$, the density operator is not k -separable.*

This corollary is useful especially in detecting entanglement of n qubits under the condition that all possible splittings are considered. When we find an antidiagonal element whose absolute value is larger than $(1/2)^n$, we immediately finish this task.

In this report, detection of n -qubit n -partite inseparability is concerned. We first introduce a theorem (a variant of the Laskowski-Żukowski corollary) to detect n -qubit n -partite inseparability from general off-diagonal elements, as a subtopic. Second, we will see the main topic where a new positive linear map Λ_P is defined as a single qubit operation. We find that the n -fold product $\Lambda_P^{\otimes n}$ may be used to detect n -qubit n -partite inseparability. This product map works, under some condition, for the class of entanglement that can be detected by the new theorem. In addition to these topics, we compare the partial- Λ_P map to the partial transpose and show a few examples of detecting n -qubit n -partite inseparability.

Theorem 1. *Suppose that we have an n -qubit n -partite density operator $\rho = \rho^{[1, \dots, n]}$. Then, ρ is not multiseparable if it has an (a, b) off-diagonal element c_{ab} such that $|c_{ab}| > 1/2^{h(a,b)}$, where $h(a, b)$ is the Hamming distance between a and b when expressed in binary notation.*

Proof. Let us write the Hilbert space of an n -qubit system as $\mathcal{H}^{[1]} \otimes \dots \otimes \mathcal{H}^{[n]}$ by using those for individual qubits, $\mathcal{H}^{[j]}$ ($j = 1, \dots, n$). Consider an n -qubit n -separable density operator $\rho_{\text{sep}}^{[1, \dots, n]} = \sum_i p_i \rho_i^{[1]} \otimes \dots \otimes \rho_i^{[n]}$ where $p_i \geq 0$ and $\sum_i p_i = 1$. Any density operator of a single qubit has diagonal elements [the $(0, 0)$ element and the $(1, 1)$ element] less than or equal to 1 and the antidiagonal elements [the $(0, 1)$ element and the $(1, 0)$ element] whose absolute values are less than or equal to $1/2$. Hence for $\forall i$, $\rho_i^{[1]} \otimes \dots \otimes \rho_i^{[n]}$ has (a, b) off-diagonal elements \tilde{c}_{ab} for which $|\tilde{c}_{ab}| \leq (1/2)^{h(a,b)}$ is satisfied. This is easy to notice when we recall that $h(a, b)$ is the number of different bits when a and b are compared as binary strings. Thus, $\rho_{\text{sep}}^{[1, \dots, n]}$ has (a, b) off-diagonal elements c_{ab} such that $|c_{ab}| \leq 1/2^{h(a,b)}$. By the contraposition of this relation, the proof is completed. \square

Let us look at the illustrating application of the theorem. The theorem is useful especially when we know the elements of a density operator. If we find an (a, b) element $c_{a,b}$ whose absolute value $> 1/2^{h(a,b)}$, we immediately find that entanglement is detected according to Theorem 1. Indeed, in order to detect that a density operator is not multiseparable, using the theorems is often more succinct than using $I^{[1, \dots, k-1, k+1, \dots, n]} \otimes \Lambda^{[k]}$ for several different values of k . Consider the three-qubit

density operator

$$\rho_b^{[1,2,3]} = \frac{1}{7b+1} \begin{pmatrix} b & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1+b}{2} & 0 & 0 & \frac{\sqrt{1-b^2}}{2} \\ b & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & b & 0 & \frac{\sqrt{1-b^2}}{2} & 0 & 0 & \frac{1+b}{2} \end{pmatrix} \quad (3)$$

($0 < b < 1$). It is known that this is positive under a partial transpose with respect to the first qubit (namely, the left most qubit when expressed in the binary expression). This matrix was originally introduced by Horodecki [2] on the basis of the range criterion as an inseparable density operator with positive partial transposition for a 2×4 bipartite system. In the present context, our interest is to find it not multiseparable. One of off-diagonal elements of $\rho_b^{[1,2,3]}$ is $\langle 7 | \rho_b^{[1,2,3]} | 4 \rangle = \sqrt{1-b^2}/(14b+2)$, which is larger than $1/2^{h(7,4)} = 1/4$ when $b < (\sqrt{57}-7)/4 \simeq 0.137$. Thus we find that $\rho_b^{[1,2,3]}$ is not multiseparable according to Theorem 1 if $b < 0.137$. In this example, the theorem is, of course, weaker than the range criterion with which inseparability is detected for $0 < b < 1$.

It might be of interest which PnCP map can detect entanglement that can be detected by the theorem. In the next step, we will see a positive linear map acting on a single qubit. According to Propositions 1 and 2 shown later, the n -fold product of this map works for this purpose in either the case where we focus solely on antidiagonal elements or the case where all the phase factors of individual elements are equal in the left lower side (lower than diagonal elements) [or, equivalently, in the right upper side (upper than diagonal elements)] of a density operator.

As we have mentioned, we introduce a positive linear map acting on a single qubit.

Definition. The positive linear map Λ_P for a single qubit is defined as follows:

$$\sigma \Rightarrow \Lambda_P(\sigma) = \begin{pmatrix} \frac{\langle 0|\sigma|0\rangle + \langle 1|\sigma|1\rangle}{2} & \frac{\langle 0|\sigma|1\rangle}{2} \\ \frac{\langle 1|\sigma|0\rangle}{2} & \frac{\langle 0|\sigma|0\rangle + \langle 1|\sigma|1\rangle}{2} \end{pmatrix}, \quad (4)$$

where σ is a single-qubit density operator and we use the basis $\{|0\rangle, |1\rangle\}$.

This map may be called a population-only decay map because of its definition.

Note. It is easy to find that Λ_P is a positive map for a single qubit. For a density operator

$$\sigma = \begin{pmatrix} c_{00} & x \\ x^* & c_{11} \end{pmatrix},$$

the relation $|x| \leq \sqrt{c_{00}c_{11}} \leq \frac{1}{2}$ holds. Λ_P acting on σ changes diagonal elements into $1/2$ and preserves off-diagonal elements. Therefore it is a positive map for a single qubit.

Remark (i). When we see the effect of Λ_P in the Bloch ball of a single qubit, Λ_P is a projection onto the x - y plane as illustrated in Fig. 1.

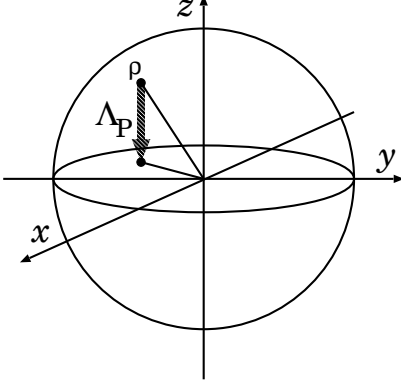


FIG. 1: The effect (shown as an arrow) of Λ_P acting on a single-qubit density operator $\rho(x, y, z) = I/2 + x\sigma_X + y\sigma_Y + z\sigma_Z$ illustrated in the Bloch ball by its coordinate (x, y, z) . Here, σ_X , σ_Y , and σ_Z are the Pauli matrices. The coordinate is changed to $(x, y, 0)$ by Λ_P .

Remark (ii). Let us represent the product of the map Λ_P acting on the k th qubit and the identity map acting on the rest of the qubits by $I^{[1, \dots, k-1, k+1, \dots, n]} \otimes \Lambda_P^{[k]}$. This product map changes the value of the $(i_1 \dots i_n, j_1 \dots j_n)$ element (here, $i_1, \dots, i_n, j_1, \dots, j_n \in \{0, 1\}$) of an Hermitian operator ξ into:

- (i) $\frac{1}{2}(\langle i_1 \dots i_{k-1} 0 i_{k+1} \dots i_n | \xi | j_1 \dots j_{k-1} 0 j_{k+1} \dots j_n \rangle + \langle i_1 \dots i_{k-1} 1 i_{k+1} \dots i_n | \xi | j_1 \dots j_{k-1} 1 j_{k+1} \dots j_n \rangle)$ when $i_k = j_k$,
- (ii) $\langle i_1 \dots i_n | \xi | j_1 \dots j_n \rangle$ when $i_k \neq j_k$.

Thus, we have

$$(I^{[1, \dots, k-1, k+1, \dots, n]} \otimes \Lambda_P^{[k]})(\rho) = \sum_{\mathbf{a}, \mathbf{c}, \mathbf{b}, \mathbf{d}} \left[\frac{1}{2}(\langle \mathbf{a} \mathbf{c} | \text{Tr}_k \rho | \mathbf{b} \mathbf{d} \rangle (\langle \mathbf{a} \mathbf{0} \mathbf{c} | \langle \mathbf{b} \mathbf{0} \mathbf{d} | + \langle \mathbf{a} \mathbf{1} \mathbf{c} | \langle \mathbf{b} \mathbf{1} \mathbf{d} |) + \langle \mathbf{a} \mathbf{0} \mathbf{c} | \rho | \mathbf{b} \mathbf{1} \mathbf{d} \rangle \langle \mathbf{a} \mathbf{0} \mathbf{c} | \langle \mathbf{b} \mathbf{1} \mathbf{d} | + \langle \mathbf{a} \mathbf{1} \mathbf{c} | \rho | \mathbf{b} \mathbf{0} \mathbf{d} \rangle \langle \mathbf{a} \mathbf{1} \mathbf{c} | \langle \mathbf{b} \mathbf{0} \mathbf{d} |) \right],$$

where $\text{Tr}_k \rho$ is the density operator after tracing out the k th qubit; $\mathbf{a} = i_1 \dots i_{k-1}$, $\mathbf{c} = i_{k+1} \dots i_n$, $\mathbf{b} = j_1 \dots j_{k-1}$, and $\mathbf{d} = j_{k+1} \dots j_n$.

Observation. Suppose that we have a multipartite density operator of n qubits,

$$\rho = \sum_i p_i \rho_i^{[1]} \otimes \dots \otimes \rho_i^{[n]}. \quad (5)$$

Then, $(I^{[1, \dots, k-1, k+1, \dots, n]} \otimes \Lambda_P^{[k]})(\rho)$ is positive. The contraposition of this relation leads to that, for a density operator ρ , if $(I^{[1, \dots, k-1, k+1, \dots, n]} \otimes \Lambda_P^{[k]})(\rho)$ is a non-positive operator, then ρ cannot be represented in the form of Eq. (5) i.e. ρ is not multipartite. Similarly, for a density operator ρ , if $(\Lambda_P^{\otimes n})(\rho)$ is a non-positive operator, then ρ is not multipartite.

As is well known, this logic was explained intensively by M. Horodecki and P. Horodecki [3]. Now we will see two remarks relating to this observation.

Remark (i). In detecting entanglement of two qubits, the partial population-only decay $I \otimes \Lambda_P$ is not so strong as the partial transpose $I \otimes \Lambda_T$, where Λ_T is the single-qubit transpose operation. For a two-qubit density operator $\rho^{[1,2]}$, we can write that

$$(I \otimes \Lambda_P)(\rho^{[1,2]}) = \frac{1}{2}[\rho^{[1,2]} + (I \otimes \Lambda_X)(I \otimes \Lambda_T)(\rho^{[1,2]})], \quad (6)$$

where Λ_X is a single-qubit NOT operation: $\Lambda_X(\sigma) = X\sigma X$ (here, σ is a 2×2 matrix and $X = |0\rangle\langle 1| + |1\rangle\langle 0|$). Thus, $(I \otimes \Lambda_P)(\rho^{[1,2]})$ is sometimes positive for a density operator having the property of negative partial transposition. It is clear that $(I \otimes \Lambda_T)(\rho^{[1,2]})$ is non-positive if $(I \otimes \Lambda_P)(\rho^{[1,2]})$ is non-positive.

For example, consider the two-qubit isotropic states [3, 19],

$$\rho_{\text{iso}}^{[1,2]} = \frac{|\phi\rangle\langle\phi| + s\mathbb{1}/4}{1+s},$$

where $|\phi\rangle \in \{\frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)\}$; $s \leq -4$ or $0 \leq s$. It is known that $\rho_{\text{iso}}^{[1,2]}$ is entangled for $0 \leq s < 2$ because $(I \otimes \Lambda_T)(\rho_{\text{iso}}^{[1,2]})$ has eigenvalues $(s-2)/(4s+4)$ and $(s+2)/(4s+4)$ (with the multiplicity of 3 for the latter one) irrespectively of $|\phi\rangle$. When we use the partial population-only decay map, we have

$$(I \otimes \Lambda_P)(\rho_{\text{iso}}^{[1,2]}) \in \left\{ \frac{\mathbb{1}}{4} \pm \frac{1}{2(1+s)}(|00\rangle\langle 11| + |11\rangle\langle 00|), \frac{\mathbb{1}}{4} \pm \frac{1}{2(1+s)}(|01\rangle\langle 10| + |10\rangle\langle 01|) \right\}.$$

We find that $(I \otimes \Lambda_P)(\rho_{\text{iso}}^{[1,2]})$ has eigenvalues $(s-1)/(4s+4)$, $(s+3)/(4s+4)$, and $1/4$ with the multiplicity of 2 irrespectively of $|\phi\rangle$. The first eigenvalue is negative for $0 \leq s < 1$. This interval in which entanglement is detected is half of that we obtain when $I \otimes \Lambda_T$ is used.

Remark (ii). It should be remarked that $\Lambda_P^{\otimes n}$ can be used for detecting entanglement. In contrast, if we use the single-qubit transpose operation Λ_T , $\Lambda_T^{\otimes n}$ is nothing but a transpose and it cannot be used for the detection of entanglement.

For example, consider an obviously entangled pure state $|\varphi\rangle = \sqrt{p}|00\rangle + \sqrt{1-p}|11\rangle$ with $0 < p < 1$. Then,

$$(I \otimes \Lambda_P)(|\varphi\rangle\langle\varphi|) = \begin{pmatrix} \frac{p}{2} & 0 & 0 & \sqrt{p(1-p)} \\ 0 & \frac{p}{2} & 0 & 0 \\ 0 & 0 & \frac{1-p}{2} & 0 \\ \sqrt{p(1-p)} & 0 & 0 & \frac{1-p}{2} \end{pmatrix}.$$

This has four eigenvalues, $p/2, (1-p)/2, 1/4 \pm (\sqrt{-12p^2 + 12p + 1})/4$. It has a negative eigenvalue

when $0 < p < 1$. Similarly, for the same pure state,

$$(\Lambda_P \otimes \Lambda_P)(|\varphi\rangle\langle\varphi|) = \begin{pmatrix} \frac{1}{4} & 0 & 0 & \sqrt{p(1-p)} \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ \sqrt{p(1-p)} & 0 & 0 & \frac{1}{4} \end{pmatrix}.$$

This has four eigenvalues, $1/4, 1/4, 1/4 \pm \sqrt{p-p^2}$. It has a negative eigenvalue when $\frac{1}{2} - \frac{\sqrt{3}}{4} < p < \frac{1}{2} + \frac{\sqrt{3}}{4}$. Indeed, the entanglement detection using $\Lambda_P \otimes \Lambda_P$ does not work for the intervals $0 < p < \frac{1}{2} - \frac{\sqrt{3}}{4} \simeq 0.067$ and $1 > p > \frac{1}{2} + \frac{\sqrt{3}}{4} \simeq 0.933$, but still works for the wide range of p .

Using $\Lambda_P^{\otimes n}$, we prove the following propositions using lemmas shown later.

Proposition 1. *For an n -qubit density operator $\rho = \rho^{[1, \dots, n]}$, $(\Lambda_P^{\otimes n})(\rho) \not\geq 0$ holds if ρ has an (a, b) antidiagonal element c_{ab} [i.e., $h(a, b) = n$] such that $|c_{ab}| > 1/2^n$ where $h(a, b)$ is the Hamming distance between a and b when expressed in binary notation and $a, b \in \{0, \dots, 2^n - 1\}$.*

Proof. Consider the n -qubit density operator ρ . It is assumed that it has some off-diagonal element c_{ab} such that $h(a, b) = n$ and $|c_{ab}| > 1/2^n$. Then apply the operator $\Lambda_P^{\otimes n}$ to ρ . This leads to that

$$\sigma = (\Lambda_P^{\otimes n})(\rho) = \text{diag}(1/2^n, \dots, 1/2^n) \\ + \text{off-diagonal elements.}$$

Note that the (a, b) element is unchanged by $\Lambda_P^{\otimes n}$ when $h(a, b) = n$. Thus $|\langle a|\sigma|b\rangle| = |c_{ab}| > 1/2^n$. Hence σ is a non-positive operator according to Lemma 2. \square

Proposition 2. *Suppose that we have an n -qubit density operator $\rho = \rho^{[1, \dots, n]}$ such that its off-diagonal elements c_{ij} satisfy the relation $\text{Arg}(c_{ij}) = \text{Arg}(c_{kl})$ for $i > j$ and $k > l$. Then, $(\Lambda_P^{\otimes n})(\rho) \not\geq 0$ holds if ρ has an (a, b) off-diagonal element c_{ab} such that $|c_{ab}| > 1/2^{h(a, b)}$.*

Proof. Consider an n -qubit density operator ρ with off-diagonal elements c_{ij} such that $\text{Arg}(c_{ij}) = \text{Arg}(c_{kl})$ for $i > j$ and $k > l$. It is assumed that it has some off-diagonal element c_{ab} such that $|c_{ab}| > 1/2^{h(a, b)}$. Then apply the operator $\Lambda_P^{\otimes n}$ to ρ . This leads to

$$\sigma = (\Lambda_P^{\otimes n})(\rho) = \text{diag}(1/2^n, \dots, 1/2^n) \\ + \text{off-diagonal elements.}$$

According to Lemma 1, $|\langle a|\sigma|b\rangle| \geq |c_{ab}|/2^{n-h(a, b)} > 1/2^n$. Hence σ is a non-positive operator according to Lemma 2. \square

These propositions illustrate an ability of $\Lambda_P^{\otimes n}$: it fails to map a density matrix to a non-positive Hermitian matrix for some obviously entangled state, as we have seen, but it succeeds for some state positive under a partial transpose, such as the state of Eq. (3) for $b < 0.137$.

In proving above propositions, we have utilized the following lemmas.

Lemma 1. *Suppose that an n -qubit density operator ρ has (i, j) elements c_{ij} such that $\text{Arg}(c_{ij}) = \text{Arg}(c_{kl})$ for $i > j$ and $k > l$. Consider the (a, b) element c_{ab} of ρ . Then, $(\Lambda_P^{\otimes n})(\rho)$ has the (a, b) element \tilde{c}_{ab} such that $|\tilde{c}_{ab}| \geq |c_{ab}|/2^{n-h(a, b)}$.*

Proof. Consider binary bits x and y ; binary strings \mathbf{a} and \mathbf{b} with bit length $k-1$; binary strings \mathbf{c} and \mathbf{d} with bit length $n-k$. Consider the element of ρ , $\langle \mathbf{a} \mathbf{x} \mathbf{c} | \rho | \mathbf{b} \mathbf{y} \mathbf{d} \rangle$. Let ρ evolve under $I^{[1, \dots, k-1, k+1, \dots, n]} \otimes \Lambda_P^{[k]}$. Then, $\langle \mathbf{a} \mathbf{x} \mathbf{c} | (I^{[1, \dots, k-1, k+1, \dots, n]} \otimes \Lambda_P^{[k]})(\rho) | \mathbf{b} \mathbf{y} \mathbf{d} \rangle$ is equal to $\frac{1}{2}(\langle \mathbf{a} \mathbf{0} \mathbf{c} | \rho | \mathbf{b} \mathbf{0} \mathbf{d} \rangle + \langle \mathbf{a} \mathbf{1} \mathbf{c} | \rho | \mathbf{b} \mathbf{1} \mathbf{d} \rangle)$ if $x = y$; otherwise it is equal to $\langle \mathbf{a} \mathbf{x} \mathbf{c} | \rho | \mathbf{b} \mathbf{y} \mathbf{d} \rangle$. Note that if $\mathbf{a} \mathbf{0} \mathbf{c} > \mathbf{b} \mathbf{0} \mathbf{d}$, then $\mathbf{a} \mathbf{1} \mathbf{c} > \mathbf{b} \mathbf{1} \mathbf{d}$; if $\mathbf{a} \mathbf{0} \mathbf{c} < \mathbf{b} \mathbf{0} \mathbf{d}$, then $\mathbf{a} \mathbf{1} \mathbf{c} < \mathbf{b} \mathbf{1} \mathbf{d}$. Hence, $I^{[1, \dots, k-1, k+1, \dots, n]} \otimes \Lambda_P^{[k]}$ does not change the argument but changes the absolute value of the $(\mathbf{a} \mathbf{x} \mathbf{c}, \mathbf{b} \mathbf{y} \mathbf{d})$ element by the factor $\geq 1/2$ in the case of $x = y$, because of equal argument values in the $(\mathbf{a} \mathbf{0} \mathbf{c}, \mathbf{b} \mathbf{0} \mathbf{d})$ and $(\mathbf{a} \mathbf{1} \mathbf{c}, \mathbf{b} \mathbf{1} \mathbf{d})$ elements. Thus, if we continue to apply $I^{[1, \dots, k-1, k+1, \dots, n]} \otimes \Lambda_P^{[k]}$ from $k = 1$ to $k = n$, then for each k , reduction in the absolute value by the factor $\geq 1/2$ occurs if $x = y$. Therefore $|c_{ab}|/2^{n-h(a, b)}$ is a lower bound of $|\tilde{c}_{ab}|$. \square

Lemma 2. *Suppose that an Hermitian operator σ has an (a, b) element $s_{ab} \in \mathbb{C}$ ($a \neq b$) such that $|s_{ab}| > 1/2^n$ and has diagonal elements $s_{ii} = 1/2^n$ for $\forall i$. Then, σ is a non-positive operator.*

Proof. Let us use the vector $|v\rangle = |a\rangle - (s_{ab}^*/|s_{ab}|)|b\rangle$. Then we have $\langle v|\sigma|v\rangle = s_{aa} + s_{bb} - |s_{ab}| - |s_{ab}| = 2(1/2^n - |s_{ab}|) < 0$. \square

To summarize, we have shown a variant of the Laskowski-Żukowski corollary that can be used to detect the n -qubit n -partite inseparability from any one of off-diagonal elements of a density operator. To detect the n -qubit n -partite inseparability, we have also introduced and investigated a new positive map Λ_P acting on a single qubit and its n -fold product, $\Lambda_P^{\otimes n}$.

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